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On the solution of parabolic and hyperbolic inverse heat conduction problems

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Abstract

A parabolic and hyperbolic numerical formulation of the boundary inverse heat conduction problems is considered in this paper. The control volume algorithm is combined with a digital filter method to estimate temperature and heat flux values on a surface of a body based on the temperature measurement inside the body. Numerical experiments were carried out to obtain the best value of the non negative coefficient of the hyperbolic equation using noisy and smoothed input data. The accuracy of the method is verified by comparison with a direct (analytical) solution of the problem. The influence of the relatively high noise into measurement data is studied. © 1998 Elsevier Science Ltd. All rights reserved.

Nomenclature

a thermal diffusivity
c specific heat
f smoothed measured value of temperature at time t
L number of future (or past) temperatures
l width
n_x number of nodes
q heat flux
T nodal temperature
T_0 initial temperature
t time
x spatial variable.

Greek symbols

Δt time step
β non negative coefficient
Δx control volume dimension
δ absolute error
ρ density
λ thermal conductivity
τ random values
ξ mean square error
Ω mean relative error.

Subscripts

i grid space number
k time index
K number of time steps
meas measured temperature.

1. Introduction

Over the last few years, much interest has been directed towards the use of inverse techniques for solving different engineering problems that cannot be described mathematically by direct methods. That situation occurs when all the required data to solve a direct problem or to obtain a reliable direct solution are not available. Inverse problem can be defined as a problem where all results are found when a part of them is known and some boundaries or reasons may remain unknown. Such a problem is much more difficult to solve than the direct one. The reason for this is that it is usually ill-posed [1], i.e. it is very sensitive to the measurement errors. To obtain stable results, special numerical techniques should be used. Examples of application of inverse problems include [1, 2, 3]:

(a) industrial process controlling;
(b) melting, ablation and freezing processes;
(c) designing and controlling nozzles, etc.;
(d) nuclear technology.

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Many methods are proposed in the literature for solving inverse problems. Examples include iterative gradient methods, optimization algorithms, regularization methods, function specification methods, dynamic programming methods, filtering methods, space marching method, etc. Apparently inverse problem solver needs efficient direct models and high knowledge with the stabilization methods. The main objectives of this work are

1. To draw some conclusions about the use of space marching methods for solving boundary Inverse Heat Conduction Problems (IHCPs),
2. To study the influence of all parameters on the accuracy of the algorithm.

This work differs from the other works that use space marching methods as follows:

1. A control volume method is used for discretization of the problem. This allows to solve much more complicated problem, e.g. nonlinear composite body problems.
2. A nonlinear numerical formulation of Weber method [4] is extended to a relatively high noisy input data. A digital filter method is proposed for smoothing noisy data.
3. In contrast to many filter methods the presented digital filter method does not need any information about the beginning and the end of the process. Also, any number of future and past temperatures can be easily used for smoothing noisy data.

2. Analysis

A one-dimensional body subjected to unknown heat flux and surface temperature at $x = l$ is illustrated in Fig. 1. The governing equation for the problem is

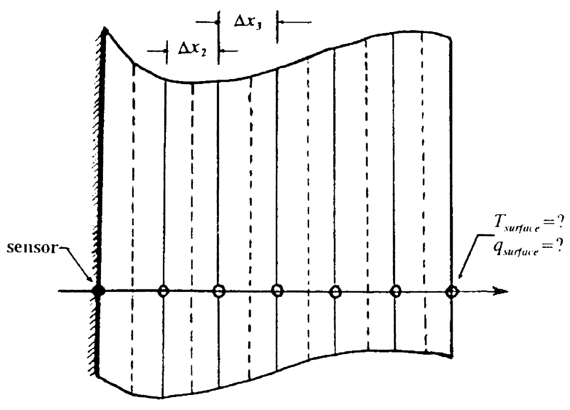

Fig. 1. Numerical discretization for one-dimensional control volume IHCP.

$$\rho(T)c(T)\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\lambda(T)\frac{\partial T(x,t)}{\partial x} \quad (1)$$

with the following boundary conditions

$$\frac{\partial T(x,t)}{\partial x} = 0 \quad \text{at } x = 0 \quad \text{and} \quad t > 0 \quad (2)$$

$$q(x,t) = -\lambda(T)\frac{\partial T(x,t)}{\partial x}$$
$$\text{is to be estimated at } x = l \wedge t > 0 \quad (3)$$

and the initial condition

$$T(x,0) = T_0 \quad 0 > x > l. \quad (4)$$

The additional condition for solving inverse problem is

$$T(0,t_k) = T_{\text{exact}}(t_k) \quad \text{for } x = 0 \wedge k = 1,2,\ldots,K \quad (5)$$

where ρ denotes the density, c is the specific heat, t is the time, T is the temperature, λ is the conductivity, T_0 is the uniform initial temperature and T_{exact} is the exact measured temperature taken from the solution of direct problem. The subscript k denotes the time index and K denotes the total number of time steps.

To solve this problem, eqn (1) should be discretized. The discretization can be done in many ways using Finite Difference Method (FDM) or Finite Element Method (FEM). In this work we adopted a Control Volume Method (CVM) in which the conservation laws applied directly over a finite size control volume. Thus, the calculation domain should be divided into finite control volumes to integrate the governing eqn (1) for each of them. Then, the finite form of the described equation can be written easily after defining the scheme of difference approximation for the time. Examples of different schemes are backward difference, Crank–Nicolson and central difference. Using the central difference scheme for approximating the time derivative we obtain the following equations:

For $i = 1$

$$\rho_{i,k}c_{i,k}\frac{\Delta x_i}{2}\frac{T_{i,k+1}-T_{i,k-1}}{2\Delta t} = \lambda_{i,k}\frac{T_{i+1,k}-T_{i,k}}{\Delta x_i} \quad (6)$$

solving eqn (6) for $T_{i+1,k}$ one obtains

$$T_{i+1,k} = \frac{\Delta x_i^2}{4a_{i,k}\Delta t}(T_{i,k+1}-T_{i,k-1})+T_{i,k} \quad (7)$$

where a is the thermal diffusivity ($a = \lambda/(\rho c)$).

For $1 < i < n_x$

$$\rho_{i,k}c_{i,k}\frac{\Delta x_{i-1}+\Delta x_i}{2}\frac{T_{i,k+1}-T_{i,k-1}}{2\Delta t}$$
$$= \lambda_{i,k}\frac{T_{i+1,k}-T_{i,k}}{\Delta x_i}+\lambda_{i-1,k}\frac{T_{i-1,k}-T_{i,k}}{\Delta x_{i-1}} \quad (8)$$

solving eqn (8) for $T_{i+1,k}$ one obtains

$$T_{i+1,k} = \frac{\Delta x_{i-1}\Delta x_i+\Delta x_i^2}{4a_{i,k}\Delta t}(T_{i,k+1}-T_{i,k-1})$$

$$+\left[\frac{\lambda_{i-1,k}}{\lambda_{i,k}}\frac{\Delta x_i}{\Delta x_{i-1}}+1\right]T_{i,k}-\frac{\lambda_{i-1,k}}{\lambda_{i,k}}\frac{\Delta x_i}{\Delta x_{i-1}}T_{i-1,k} \quad (9)$$

For $i = n_x$

$$\rho_{i,k}c_{i,k}\frac{\Delta x_{i-1}}{2}\frac{T_{i,k+1}-T_{i,k-1}}{2\Delta t}=\lambda_{i-1,k}\frac{T_{i-1,k}-T_{i,k}}{\Delta x_{i-1}}-q_{i,k} \quad (10)$$

solving eqn (10) for $q_{i,k}$ one obtains

$$q_{i,k}=\frac{\lambda_{i-1,k}}{\Delta x_{i-1}}(T_{i-1,k}-T_{i,k})-\frac{\rho_{i,k}c_{i,k}\Delta x_{i-1}}{4\Delta t}(T_{i,k+1}-T_{i,k-1}). \quad (11)$$

The calculation starts at node $i = 1$ and at $k = 1, 2, \ldots, K-1$. This means that there is no solution at $k = K$ because the temperature at $K+1$ is not defined. After completing the calculation at $i = 1$ we start the calculation at $i = 2$ and at $k = 1, 2, \ldots, K-2$ and so on. Therefore, the temperature should be taken at several additional steps (depending on the number of the nodes) to define the unknown at $k = K$. For example the measurement of temperatures at $K+1$ time steps is needed to find $T_{2,K}$. Notice the difference between the equation of the heat flux obtained from the heat balance eqn (11) and that obtained from a simple finite difference scheme ($q_{i,k} = \lambda^*_{i,k}(T_{i,k}-T_{i-1,k})/\Delta x_{i-1}$).

3. Stabilization of the inverse algorithm

Inverse heat conduction problem is usually ill-posed from the mathematical point of view because the internal temperature response is damped and lagged with respect to the surface temperature. In other words the error into the results is usually greater than the error into the input data and the solution may be oscillating. Therefore, the solution may be useless when real data are used. The better the algorithm, the weaker the sensitivity to measurement errors. Many methods are presented and discussed for stabilizing the results. An extensive review can be found in Refs [1, 3]. Beck [5] recognized that the future temperature information can be used for stabilizing the numerical results. That is because measurements taken in the past will extend into the future. Different methods using future temperatures are presented by Weber [4] and Beck et al. [1, 6]. In this work two stabilization methods will be discussed:

3.1. Digital smoothing filters

The idea of filtering is simply to replace the nodal temperature by a combination of a set of neighbouring data including itself. This has the effect of reducing the noise of the data. The filtering can be used to smooth the data spatially and/or along the time. For one-dimensional problem there is no need to smooth the data spatially. Different kinds of filters were used to stabilize the inverse algorithms such as Hanning, triangular, Gaussian and Tukey filters [7–9].

In this work a Gram orthogonal polynomial method [10, 11] with a moving averaging filter window is proposed for smoothing the noisy data. This method is based on a least square approximation. This method does not need any information about the beginning and the end of the process. This is suitable for use in on-line methods of analysis. Application of digital filtering to a series of equally spaced seven data points is shown in Fig. 2. Digital filter replaces each data value T_i by a combination of itself and a number of adjacent nodes. Thus,

$$f_n(t_k)=\sum_{j=0}^{n}b_jP_j(k) \quad (12)$$

where $f(t_k)$ is the smoothed value of measured temperature at $t = k$. Let L is the number of points used to the left (past temperatures) and to the right (future temperatures) of the central point at time k. The subscript n refers to the total number of data points used for smoothing process ($n = 5$ for $L = 2$, $n = 7$ for $L = 3$, etc.). The parameters in eqn (12) can be calculated as

$$b_j=\frac{\sum_{k=-L}^{L}T(k)P_j(k)}{\dfrac{(2L+j+1)!(2L-j)!}{(2j+1)[(2L)!]^2}} \quad (13)$$

and

$$P_j(k)=\frac{\sum_{m=0}^{j}(-1)^{m+j}(m+j)^{[2m]}(L+k)^{[m]}}{(m!)^2(2L)^{[m]}} \quad (14)$$

where T denotes the measured values of temperature. Thus, we compute $f(t)$ for each time step as the average from $T_{(t-L)}$ to $T_{(t+L)}$. This is sometimes called moving window average. Notice that $x^{[m]} = x(x-1)(x-2)\ldots(x-m+1)$ and $m = 1, 2, \ldots$

The following equation can be derived (see Appendix) to find the smoothed measured value of temperature at

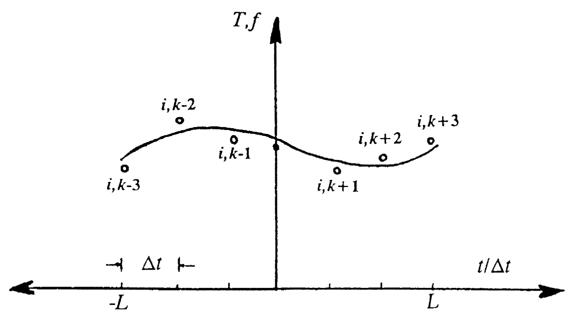

Fig. 2. Smoothing of the measured temperatures using seven points averaging filter.

the center time k considering three past times ($k-3, k-2, k-1$) and three future times ($k+1, k+2, K+3$):

$$f_3(t_k) = \tfrac{1}{21}[-2T_{i,k-3}+3T_{i,k-2}+6T_{i,k-1} + 7T_{i,k}+6T_{i,k+1}+3T_{i,k+2}-2T_{i,k+3}]. \quad (15)$$

The whole time interval is moving one time step for the next calculation and so on. At the first steps the past temperatures can be set to be equal to the initial temperature.

Clearly eqns (12)–(14) can be used to consider different past and future temperatures. The use of seven to 11 data points is enough to obtain a good approximation. There is an optimal relation between the size of time step and considering future temperatures for a given set of measurement errors [1, 3]. It is recommended to use many future temperatures when Δt is small. The choice of the time step for solving one-dimensional problem is not troublesome [1, 12]. The temperature variation between two measurements should be large enough to see variation of temperature and should be greater than measurement errors.

It is worth mentioning that time derivatives of the temperature data can be derived from eqns (12)–(14) after a mathematical manipulation. This will be helpful for solving the exact solution of Bruggraf [13] using real data.

3.2. Hyperbolic equation for solving IHCP

Weber [4] has proposed the use of hyperbolic differential heat conduction problem for governing IHCP instead of the parabolic equation. The use of hyperbolic equation for solving direct problem is seldom and it is used for solving rapid heat conduction process in metal. Morse and Feshbach [14] have proposed it as a better model of heat conduction problem because it does not require instantaneous transfer of heat. The hyperbolic equation has the following form

$$\beta\rho(T)c(T)\frac{\partial^2 T(x,t)}{\partial t^2}+\rho(T)c(T)\frac{\partial t(x,t)}{\partial t} = \frac{\partial}{\partial x}\lambda(T)\frac{\partial T(x,t)}{\partial x} \quad (16)$$

is subjected to the eqns (2)–(5). It can be shown from eqn (16) that the influence of the additional term depends on the value of the non-negative coefficient β. This term can be negligible when very small value of β is used and for $\beta = 0$ the eqn (1) is yield. Physically, β may represent the time of relaxation (it can be defined as the ratio of the thermal diffusivity to the speed of conducting heat inside a body). A numerical experiment should be used to obtain the optimal value of β to improve the stability of IHCP. The numerical methods for hyperbolic equation are efficient and accurate [3]. Using CVM the discretization equations for the hyperbolic equation can be written as follows:

For $i = 1$

$$T_{i+1,k} = \frac{\beta}{2a_{i,k}}\frac{\Delta x_i^2}{\Delta t^2}(T_{i,k-1}-2T_{i,k}+T_{i,k+1}) + \frac{\Delta x_i^2}{4a_{i,k}\Delta t}(T_{i,k+1}-T_{i,k-1})+T_{i,k}. \quad (17)$$

For $1 < i < n_x$

$$T_{i+1,k} = \frac{\beta}{a_{i,k}}\frac{\Delta x_{i-1}\Delta x_i+\Delta x_i^2}{2\Delta t^2}(T_{i,k-1}-2T_{i,k}+T_{i,k+1}) + \frac{\Delta x_{i-1}\Delta x_i+\Delta x_i^2}{4a_{i,k}\Delta t}(T_{i,k+1}-T_{i,k-1}) + \left[\frac{\lambda_{i-1,k}}{\lambda_{i,k}}\frac{\Delta x_i}{\Delta x_{i-1}}+1\right]T_{i,k} - \frac{\lambda_{i-1,k}}{\lambda_{i,k}}\frac{\Delta x_i}{\Delta x_{i-1}}T_{i-1,k}. \quad (18)$$

For $i = n_x$

$$q_{i,k} = \frac{\lambda_{i-1,k}}{\Delta x_{i-1}}(T_{i-1,k}-T_{i,k}) - \frac{\rho_{i,k}c_{i,k}\Delta x_{i-1}}{4\Delta t}(T_{i,k+1}-T_{i,k-1}) - \frac{\beta\rho_{i,k}c_{i,k}\Delta x_{i-1}}{2\Delta t^2}(T_{i,k-1}-2T_{i,k}+T_{i,k+1}). \quad (19)$$

Weber did not show how small should be the coefficient β. A value smaller than 0.01 was used to obtain good results when the data were perturbed at each time step by uniformly disturbed random errors between $\pm 1\%$. The method of Weber was extended for solving a two-dimensional heat conduction problem [15] when an over specified (overdetermined) problem was solved. An overdetermined problem was formulated similar to a direct problem. The boundary and initial conditions were known as well as an additional condition that was the measurement of temperatures at several internal nodes. Huange et al. [15] used $\beta = 0.0001$ and the additional condition improved the accuracy of their calculations. The solution of the problem becomes much more complicated when one boundary condition remains unknown.

4. Test cases

To evaluate the accuracy of the method the exact temperature data are needed. The exact temperature data are generated by solving analytically the following direct problem. A plate of width l is considered (see Fig. 1). It is at a uniform initial temperature and exposed to the convective heat transfer at $x = l$ and insulated at $x = 0$. The analytical solution for this problem is:

$$T(x,t) = 2T(x,0)\sum_{m=1}^{\infty}\frac{\sin\mu_m\cos\dfrac{\mu_m x}{l}}{\mu_m\left(1+\dfrac{\sin 2\mu_m}{2\mu_m}\right)}$$

$$*\exp\left[-\left(\frac{\mu_m}{l}\right)^2 at\right] \quad (20)$$

where μ_m is the root of the following equation

$$\mu \tan \mu = Bi \quad (21)$$

where Bi is the Biot number. The first twelve eigenvalues of the eqn (21) are given in Table 1 for $Bi = 1.5$. The discrete heat flux is calculated from the heat balance equation using backward scheme for approximating the time derivative.

To make the solution more realistic simulated measurement data T_{meas} are generated by introducing additive errors to the exact measurement data T_{exact} according to the following equation.

$$T_{\text{meas}}(i,k) = T_{\text{exact}}(i,k) + \delta\tau \quad (22)$$

where δ is a maximum absolute error and τ is obtained using a random number generator of a uniform distribution within the interval $[-1, 1]$. Such a way of adding random errors creates very severe conditions for an inverse method [3].

Example 1: A steel plate ($\lambda = 50$ W m^{-2} °C, $a = 1.327$E-05 m^2 s^{-1}) of 0.035 m width initially at 500°C is insulated at $x = 0$ and subjected to the boundary condition of the 2nd kind at $x = l$. A thermocouple is attached to the plate at $x = 0$. The measured values of temperature are calculated from eqn (20). Find the surface temperature and heat flux along the time when $\Delta t = 10$ s.

Solution: The convective coefficient is assumed known only when solving direct problem in order to prepare data for solving inverse problem. The exact temperature data at $x = 0$ (for 22 time steps) are summarized in Table 2. The calculation domain is divided into 7 equally spaced control volumes and 30 time steps are considered to solve an inverse problem for 24 steps. The results of Example 1 are shown in Fig. 3 (for surface temperature) and Fig. 4 (for heat flux). The following conclusions can be drawn:

1. The simulated surface temperature is in a good agreement with the exact solution. Accuracy of the results was also very good when backward scheme for approximating the time derivative is used. The reason of using central difference scheme is the lower sensitivity for the noisy data.
2. The estimated heat flux agrees well with the exact value except for the first two time steps. That is because of the damping and lagging effect.
3. There is no need to combine the algorithm with the stabilization methods.

When the problem is solved using a wrong value of initial temperature, the accuracy of the results is increased along the time. This is another advantage of the proposed algorithm.

Example 2: For the same plate as in Example 1 calculate the surface heat flux and surface temperature after disturbing the input data with random errors between $\pm 5\%$ according to eqn (22).

(a) use digital filter method of seven points approximation to smooth the measured data;
(b) use the hyperbolic form of the inverse heat conduction problem;
(c) use the hyperbolic equation with the smoothed values of measured temperatures.

Solution: The obtained values of the noisy data using eqn (22) are given in Table 2. We can see that eqn (22) created very noisy data. Figure 6 shows heat flux results when using noisy data directly for solving the problem. The results are oscillating and unacceptable. Values of relative errors for heat flux $(q_{\text{exact}} - q_{\text{inverse}})/q_{\text{exact}}$ are given in Table 3. We can see measurement errors within an interval $\pm 5\%$ can lead to errors more than $\pm 50\%$ into the results. This shows clearly why inverse problem is difficult to solve. Thus, the results with using real (noisy) data requires an efficient stabilization method.

The influence of using the proposed digital filter method on smoothing the data is shown in Table 2. Results of estimation of both temperature distribution are shown in Fig. 5 and Fig. 6, respectively. The following conclusions can be drawn:

1. Stability and the accuracy of the results for both surface temperature and surface heat flux is improved greatly in comparison with that obtained using noisy data.
2. Accuracy of the surface temperature is increased along the time and after several steps very good agreement between exact and estimated results is obtained.
3. Accuracy of the results for the surface heat flux is acceptable but it is more sensitive to the measurement errors than the surface temperature especially for the last time steps.

Table 1
Roots of eqn (21)

m	1	2	3	4	5	6	7	8	9	10	11	12
μ	0.9882	3.5422	6.5097	9.5801	12.684	15.833	18.928	22.059	25.192	28.327	31.463	34.60

Table 2
Temperature input data

Time (s)	Exact data (analytical solution)	Noisy data eqn (22)	Smoothing values of temperature measurement	
			seven points averaging filter	eleven points averaging filter
10	493.587	492.066	487.644	477.771
20	460.239	467.398	458.129	452.710
30	418.272	412.208	420.399	421.322
40	377.370	372.473	378.174	384.609
50	339.766	347.065	340.574	346.433
60	305.729	306.844	309.438	309.139
70	275.058	279.948	279.266	275.893
80	247.447	247.485	248.078	249.162
90	222.607	223.622	221.065	222.645
100	200.260	194.065	197.799	197.696
110	180.156	176.391	177.050	177.816
120	162.076	163.809	158.247	159.134
130	145.800	139.372	144.332	142.514
140	131.163	128.972	129.403	128.651
150	117.996	119.966	114.611	116.311
160	106.150	101.132	105.123	104.367
170	95.494	91.704	93.910	93.645
180	85.907	89.585	84.086	84.788
190	77.283	73.639	76.721	75.727
200	69.524	68.492	68.445	67.716
210	62.545	61.140	60.336	61.668
220	56.266	54.528	55.045	55.482

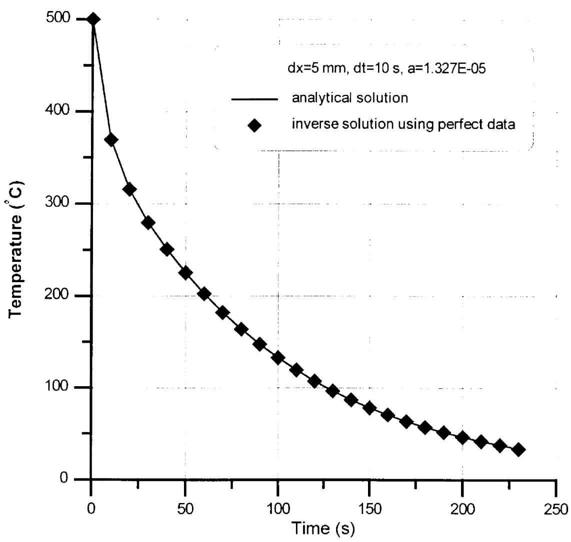

Fig. 3. Temperature distribution at the surface node.

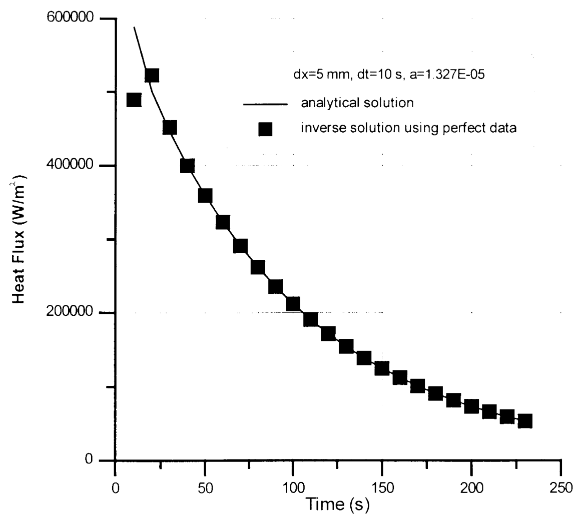

Fig. 4. Heat flux values at the surface node.

In the 2nd part of the example the noisy data are used to solve the hyperbolic problem. To solve eqns (17)–(19) we need the value of β. Weber used a very small value of 0.01 multiplied directly by the second derivative of the temperature. When we use $\beta = 0.01$ for solving the presented example, additional term of hyperbolic equation has no influence on the solution. Therefore, we propose the following parameter ψ as a criterion for controlling the algorithm

Table 3
Relative errors in the surface heat flux

Time	10	20	30	40	50	60	70	80	90	100	110	120
%	19.66	−19.14	0.40	28.30	−7.86	−8.57	−3.52	−14.65	−11.71	8.87	19.95	−27.45
Time	130	140	150	160	170	180	190	200	210	220	230	240
%	12.51	26.99	−43.57	12.63	53.14	−47.22	−12.80	31.13	−14.93	20.43	22.27	20.01

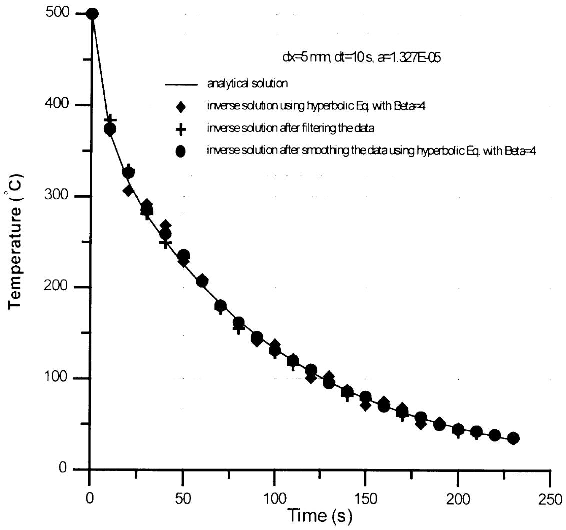

Fig. 5. Temperature distribution at the surface node using noisy data.

$$\psi = \frac{\beta}{a_i}\left(\frac{\Delta x_i}{\Delta t}\right)^2. \tag{23}$$

The above parameter considers the influence of step size, thermal properties and β. Good results are obtained for the value of ψ in the range 0.02 and 0.09. Figures 5 and 7 show the results when $\beta = 4$. Results of the surface heat flux calculations can be acceptable but the stability is worse in comparison with that obtained in the first part of the example.

Figures 5 and 8 show the results when eqns (17)–(19) are solved using smoothed values of noisy data. The following conclusions can be summarized:

1. Because the surface temperature results are in a good agreement with the results of the direct problem for all cases therefore, the influence of such a combination was minimum.
2. Accuracy and the stability of the surface heat flux are improved. The agreement between estimated and exact heat flux values is very good. This makes the proposed solution method of much interest for us. Results for different combinations will be shown in Example 3.

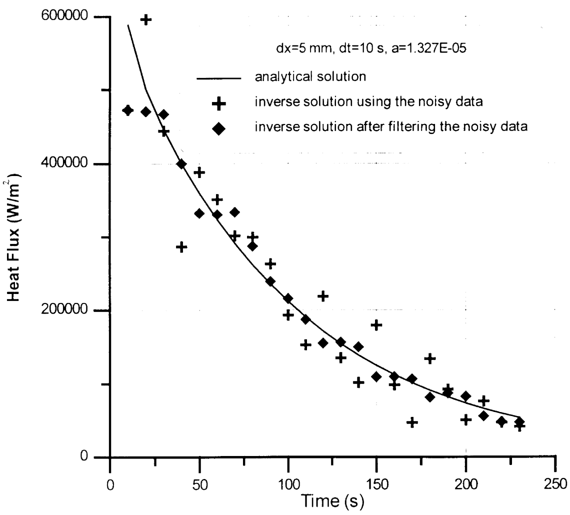

Fig. 6. Heat flux values at the surface node.

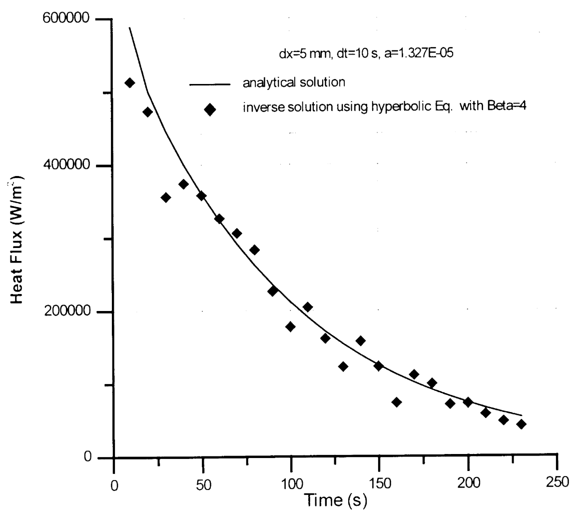

Fig. 7. Heat flux values at the surface node.

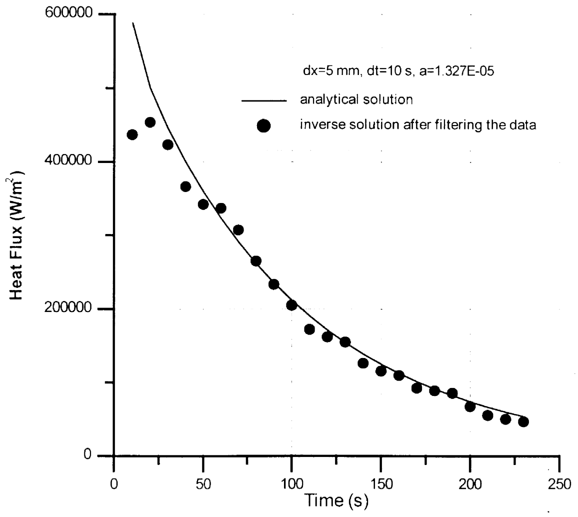

Fig. 8. Heat flux values at the surface node using hyperbolic equation with $\beta = 4$.

Because of the simplicity of considered smoothing procedure we can also smooth the results but for the presented examples there is no need to post-filtering the results.

Example 3: For Example 2 show the influence of β into the results. Use $\beta = 0$, 0.3, 4, 5 and 10. Calculate both mean relative error and mean square error.

Solution: The mean square error (ξ) is

$$\xi = \sqrt{\frac{1}{K}\sum_{k=1}^{K}(q^k_{\text{estimated}} - q^k_{\text{exact}})^2} \tag{24}$$

where q_{exact} is the exact value of heat flux taken from the solution of the direct (analytical) problem. The mean relative error (Ω) is given by:

$$\Omega = \frac{1}{K}\sum_{k=1}^{K}\frac{|q^k_{\text{estimated}} - q^k_{\text{exact}}|}{q^k_{\text{exact}}}. \tag{25}$$

Results of calculations are presented in Tables 4 and 5. The following conclusions can be drawn:

1. With using noisy data to solve eqns (17)–(19) best results are obtained at $\psi = 0.0752$ ($\beta = 4$). Both mean relative and mean square errors are reduced.
2. When the smoothed data are used to solve eqns (17)–(19) the mean relative errors are reduced for all cases and the values of mean square error depend on ψ.
3. The best results are obtained when ψ varied between 0.0188 ($\beta = 1$) and 0.094 ($\beta = 5$). We see ψ has small values. This maintains the numerical stability and the heat conduction nature of the problem.
4. The accuracy of the results is reduced again when we use $\beta > 5$.

Example 4: For Example 3 with $\beta = 2$:

(a) estimate the surface heat flux considering different number of future temperatures for smoothing noisy input data. Use three and five future temperatures;
(b) calculate the mean relative error and mean square error for each case.

Solution: Smoothed values of noisy data using a set of seven data points is calculated from eqn (15) while the smoothed values of noisy data using a set of eleven data points can be calculated from the following equation (derived from eqns (12)–(14)):

$$f_5(t_k) = \tfrac{1}{429}[-36T_{i,k-5} + 9T_{i,k-4} + 44T_{i,k-3} + 69T_{i,k-2} + 84T_{i,k-1} + 89T_{i,k} + 84T_{i,k+1} + 69T_{i,k+2} + 44T_{i,k+3} + 9T_{i,k+4} - 36T_{i,k+5}]. \tag{26}$$

Results of calculation for part *a* are shown in Fig. 9. The estimated surface heat flux agrees well with the exact solution of the problem. At the first steps the results were better with using seven points averaging filter. This can be shown clearly when the second part of the example is solved (23 time steps are considered). Values of mean relative and mean square errors are reported in Table 6. We see from Table 6 that the mean relative error is reduced while the mean square error is increased. Therefore, the over smoothing data is not recommended because the stability increases but this leads to reduce the accuracy (the variance increases). However, considering future temperatures depends on the size of time step.

5. Conclusions

This work provides a good insight on using control volume formulation for solving inverse heat conduction problems. The main conclusions can be summarized as follows:

Table 4
The influence of ψ on solution using noisy data taken from Table 2

ψ	0.0 ($\beta = 0$)	0.00564 ($\beta = 0.3$)	0.0564 ($\beta = 3$)	0.0752 ($\beta = 4$)	0.094 ($\beta = 5$)	0.188 ($\beta = 10$)
ξ (%)	20.340	18.951	11.346	11.017	11.960	16.18
Ω (W m^{-2})	10101	9368.6	6303.3	6275.4	6788.1	12961.9

Table 5
The influence of ψ on solution using smoothed data taken from Table 2

ψ	0. ($\beta = 0$)	0.00564 ($\beta = 0.3$)	0.0188 ($\beta = 1$)	0.0376 ($\beta = 2$)	0.0564 ($\beta = 3$)	0.0752 ($\beta = 4$)	0.094 ($\beta = 5$)	0.188 ($\beta = 10$)
ξ (%)	9.60	7.67	7.08	6.84	7.202	7.53	7.93	12.24
Ω (W m^{-2})	6005	5955	5963	6224	6705	7348	8103	12552

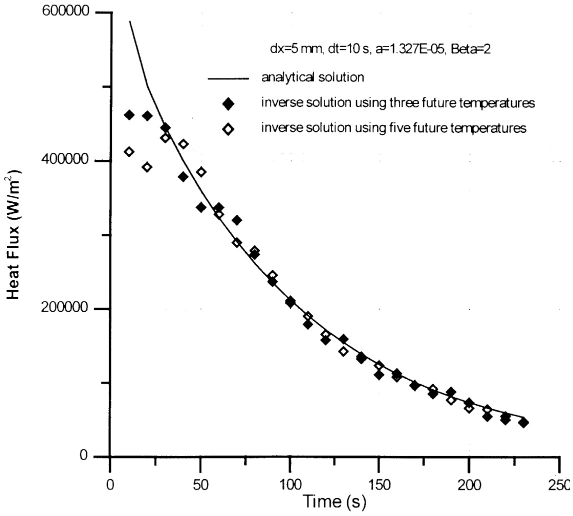

Fig. 9. Estimated surface heat flux values using different future temperatures for smoothing the data.

1. Accuracy and the stability of results are improved when a hyperbolic form of heat conduction problem is solved using the control volume method (for discretization) and the digital filter method (for stabilization).
2. Use the proposed algorithm enables to handle a relatively high noisy input data.
3. No iteration is needed. Therefore, the computation time is very short.
4. The proposed methods can be extended for solving multi-dimensional problems. The first investigation showed good results for estimating of surface temperatures.
5. It can be used for composite bodes and for nonlinear problems.
6. It is not important to place the sensor at an insulating surface. The sensor can be placed anywhere inside the body. In such a case we need the value of heat flux at the node where the sensor is located or two interior sensors should be considered.

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Appendix

The derivation of eqn (15) is seen below. To calculate the smoothed values of measured temperature at the center of seven data points $L = 3$ and $n = j = 3$. Thus,
From eqn (12)

$$f_3(t_k) = b_0 P_0(k) + b_1 P_1(k) + b_2 P_2(k) + b_3 P_3(k) \quad \text{(a)}$$

From eqn (14)

$$P_0(k) = 1 \quad P_1(k) = \tfrac{1}{3}k$$

$$P_2(k) = \tfrac{1}{5}(k^2 - 4) \quad P_3(k) = \tfrac{1}{6}(k^3 - 7k) \quad \text{(b)}$$

where $k = -L, -L+1, \ldots, L$.
From eqn (13)

$$b_0 = \tfrac{1}{7}[T_{-3} + T_{-2} + T_{-1} + T_0 + T_1 + T_2 + T_3]$$

$$b_1 = \tfrac{1}{28}[-9T_{-3} - 6T_{-2} - 3T_{-1} + 3T_1 + 6T_2 + 9T_3]$$

$$b_2 = \tfrac{1}{84}[25T_{-3} - 15T_{-1} - 20T_0 - 15T_1 + 25T_3]$$

$$b_3 = \tfrac{1}{6}[-T_{-3} + T_{-2} + T_{-1} - T_1 - T_2 + T_3] \quad \text{(c)}$$

substituting eqn (b) and eqn (c) into eqn (a) to obtain

Table 6
The mean relative and mean square errors in the estimated surface flux

number of future and past temperatures	seven data points	eleven data points
ξ (%)	6.8407	6.34
Ω (W m^{-2})	6224.3	9222

$$f_3(t_k) = \tfrac{1}{7}[T_{-3}+T_{-2}+T_{-1}+T_0+T_1+T_2+T_3]$$
$$+\tfrac{1}{28}[-9T_{-3}-6T_{-2}-3T_{-1}+3T_1+6T_2+9T_3]k+$$
$$+\tfrac{1}{84}[25T_{-3}-15T_{-1}-20T_0-15T_1+25T_3](k^2-4)$$
$$+\tfrac{1}{6}[-T_{-3}+T_{-2}+T_{-1}-T_1-T_2+T_3](k^3-7k). \quad \text{(d)}$$

We see eqn (d) can be used to smooth the data spatially, for each value of k new equation is obtained. For one-dimensional problem there is no need for such a smoothing because the temperature is measured only at one node. Rewriting of eqn (d) at the center of the seven data points ($k = 0$) gives

$$f_3(0) = \tfrac{1}{21}[-2T_{-3}+3T_{-2}+6T_{-1}$$
$$+7T_0+6T_1+3T_2-2T_3]. \quad \text{(e)}$$

The general form of the above equation gives eqn (15). For more information see [10, 11].

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